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LETTER TO THE EDITOR

On the equivalence between action-at-a-distance and non-linear field theories: three-body forces

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Abstract. The equivalence between direct-interaction theories and non-linear field theories is discussed on a classical (non-quantum) basis. We show that to have such an equivalence three-body forces must be introduced, if there is not coupling between different interactions, in the action-at-a-distance framework. Finally a new type of Poincaré-invariant theories containing three-body forces is proposed.

We shall discuss in this letter the problem of the equivalence between *action-at-a-distance* and *non-linear field* theories in classical (i.e. non-quantum) physics. At this stage we will deal mainly with general principles, irrespectively of their physical realization in nature. To this purpose we shall consider a simple idealized experiment which has been recently suggested by Pegg (1975), although our treatment of it and the conclusions that we reach are completely different to those obtained by the above mentioned author.

Let us consider a system of three particles (1, 2, 3) where two of them (1, 2) have charge (e_1, e_2) but no mass ($m_1 = m_2 = 0$). To avoid difficulties we shall assume that both particles move with the speed of light. The third particle (3) has mass (m_3) but no charge ($e_3 = 0$). If we remain within the framework of action-at-a-distance theories the equations of motion for particle 3 will be (Chern and Havas 1973)

$$m_3\frac{\mathrm{d}^2 z_3^\alpha}{\mathrm{d}\tau_3^2} = F_{31}^\alpha + F_{32}^\alpha \tag{1}$$

where τ_3 corresponds, at this stage, to an arbitrary parametrization along the world-line of particle 3, and F_{ab} ($a, b, c = 1, 2, 3$, $a \neq b \neq c$, $a \neq c$; $\alpha, \beta, \ldots = 0, 1, 2, 3$) is the force acting on particle a due to particle b. *In the case of theories without mass–charge coupling* there will be no interaction between particle 3 and particles 1 and 2, therefore particle 3 will remain at rest in a suitable system of reference. This contrasts with the situation in a *non-linear field* theory such as, for example, Einstein's general relativity (in a *linear field* theory without mass–charge coupling particle 3 will also remain at rest). In this case the field which appears due to the electromagnetic interaction between particles 1 and 2 has an associate energy and hence it will have also an associate mass. Therefore, and due to the non-linearity of the field, there will be a gravitational interaction between the field and particle 3 which as a consequence will move. That is, there is no possible equivalence between action-at-a-distance theories and non-linear field theories if equations of type (1), in the absence of mass–charge coupling, are used in the action-at-a-distance case.

This difficulty can be overcome without introducing coupling between different interactions (mass–charge in our example) by considering either of the two following equations of motion in a direct-interaction theory:

$$m_3 \frac{d^2 z_3^\alpha}{d\tau_3^2} = F_{31}^\alpha + F_{32}^\alpha + F_{123}^\alpha \tag{2}$$

or

$$m_3 \frac{d^2 z_3^\alpha}{d\tau_3^2} = F_{123}^\alpha. \tag{3}$$

Three-body forces such as F_{123}, without coupling between different interactions but dependent on the coupling constants of the three particles for a given type of interaction are in principle perfectly admissible†. Let us represent the coupling constants by g_a (g-interaction) where a denotes the particle under consideration, and the g-dependence of the three-body forces by $F_{123}(g_1, g_2, g_3; \gamma)$, γ standing for any other kinds of dependence (kinematical for example). Obviously the fact that one or several of the g_a equals zero does not imply $F_{123}^\alpha(g_1, g_2, g_3; \gamma) = 0$; as possible examples we have

$$\frac{g_1 g_2}{g_3 + d}, \qquad \frac{g_1}{g_2 + g_3 + d}, \qquad \frac{h}{g_1 + g_2 + g_3 + d}, \qquad (d, h \text{ constants} \neq 0).$$

In fact in our thought experiment we find such situations: in an e-interaction framework we have $F_{123}^\alpha(e_1, e_2, 0; \gamma)$, and in the m-interaction $\bar{F}_{123}^\alpha(0, 0, m_3; \bar{\gamma})$. It should be pointed out, however, that coupling-constant combinations of the type $g_1 g_2 g_3$ (which as a matter of fact are the ones usually employed) lead to null three-body forces (no interaction). Another feature to be mentioned is that the absence of mass–charge coupling in F_{123} enables us to introduce a 'principle of equivalence' ('generalized principles of equivalence' in the case of non-gravitational interactions), i.e. particle 3 falls in the 'field' produced by particles 1 and 2 independently of its mass; of course it is necessary for this purpose to choose $F_{123}^\alpha(e_1, e_2, 0; \gamma)$ instead of $\bar{F}_{123}^\alpha(0, 0, m_3; \bar{\gamma})$.

Two-body forces, F_{ab}, have already been studied (Havas 1971, Cordero and Ghirardi 1973), therefore we shall not mention them here; we shall be concerned only with the three-body force terms F_{abc}. In this sense and supposing that the Lagrangians we are going to consider depend only on z_1^α, z_2^α, z_3^α, u_1^α, u_2^α, u_3^α ($u_a^\alpha \equiv dz_a^\alpha/d\tau_a$), we propose, as a possible election, to assume that the three-body force terms can be derived from a generalized Poincaré-invariant Fokker-type interaction Lagrangian in a Minkowski space–time,

$$J_{123} = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \Lambda_{123} \, d\tau_1 \, d\tau_2 \, d\tau_3 \tag{4}$$

where Λ_{123} depends, in order to have theories invariant under the Poincaré group, on

† Although we do not attempt to study this case here let us say that if we have N particles ($N > 3$), n-body forces ($n = 4, 5, \ldots, N$) are also possible. That is, the possibility that n-body forces should be introduced in a direct-interaction theory in order to make it equivalent to a non-linear field theory must be considered.

the following maximal set of independent Poincaré-scalars:

$$\begin{aligned}&(s_{12})^2, (s_{13})^2, (s_{12}\,.\,s_{13})\\&(u_1)^2, (u_2)^2, (u_3)^2, (u_1\,.\,u_2), (u_1\,.\,u_3), (u_2\,.\,u_3)\\&(s_{12}\,.\,u_1), (s_{12}\,.\,u_2), (s_{12}\,.\,u_3), (s_{13}\,.\,u_1), (s_{13}\,.\,u_2), (s_{13}\,.\,u_3),\end{aligned}\tag{5}$$

where $s^\alpha_{ab}\equiv(z^\alpha_a-z^\alpha_b)$, $a\neq b$, $(x\,.\,y)\equiv x^\beta y_\beta$, $(x)^2\equiv x^\beta x_\beta$.

After the standard variational procedure we obtain from (4)

$$F^\alpha_{123}=\int\int f^\alpha_{123}\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2;$$

the integrand f^α_{123} may be interpreted as the force acting on particle 3 due to particles 1 and 2 in the intervals $(\tau_1, \tau_1+\mathrm{d}\tau_1)$ and $(\tau_2, \tau_2+\mathrm{d}\tau_2)$, and is given by

$$\begin{aligned}f^\alpha_{123}=\Bigg[&2s^\alpha_{31}\frac{\partial}{\partial(s_{31})^2}+s^\alpha_{12}\frac{\partial}{\partial(s_{12}\,.\,s_{31})}+\sum_{a=1}^{3}u^\alpha_a\frac{\partial}{\partial(s_{31}\,.\,u_a)}\\&-\frac{\mathrm{d}}{\mathrm{d}\tau_3}\left(2u^\alpha_3\frac{\partial}{\partial(u_3)^2}+\sum_{a=1}^{2}u^\alpha_a\frac{\partial}{\partial(u_a\,.\,u_3)}+\sum_{a=2}^{3}s^\alpha_{1a}\frac{\partial}{\partial(s_{1a}\,.\,u_3)}\right)\Bigg]\Lambda_{123}.\end{aligned}\tag{6}$$

As we have more than two particles we have enough vectors available to allow for the construction of pseudoscalars in addition to the scalars (5). In this sense we can develop a theory allowing non-conservation of parity by making Λ_{123} depend also on the pseudoscalars

$$\begin{aligned}&\epsilon_{\alpha\beta\rho\delta}s^\alpha_{12}s^\beta_{13}u^\rho_1u^\delta_2\\&\epsilon_{\alpha\beta\rho\delta}s^\alpha_{12}s^\beta_{13}u^\rho_1u^\delta_3\\&\epsilon_{\alpha\beta\rho\delta}s^\alpha_{12}s^\beta_{13}u^\rho_2u^\delta_3\\&\epsilon_{\alpha\beta\rho\delta}s^\alpha_{12}u^\beta_1u^\rho_2u^\delta_3\\&\epsilon_{\alpha\beta\rho\delta}s^\alpha_{13}u^\beta_1u^\rho_2u^\delta_3\end{aligned}\tag{7}$$

where $\epsilon_{\alpha\beta\rho\delta}$ is zero if any two indices are equal, and ±1 according to $(\alpha\beta\rho\delta)$ being an even or odd permutation of the numbers (1, 2, 3, 4).

The programme to be carried out in order to develop these new action-at-a-distance theories will be, at least in the first stages, to compute the corresponding laws of conservation, to try to find exact solutions of the equations of motion and, if successful, to analyse them in order to see their physical suitability (i.e. stability of the solutions, physical situations they can represent, etc). All these points will be done elsewhere.

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